Week 8

Ex5AQ5 **Problem 1**: Let V be a vector space and $T \in \mathcal{L}(V)$. Suppose $\{U_i\}_{i\in I}$ is a collection of T-invariant subspaces of V. Prove that $\bigcap_{i\in I} U_i$ is a T-invariant subspace of V.

Sol^n: Recall that a subspace U of V is said to be T-invar. if $Tu \in U$ $\forall u \in U$.

Pick $u \in \bigcap_{i\in I} U_i$. Then $u \in U_i$ $\forall i \in I$.

Since U_i is T-invar, $Tu \in U_i$. This is true for all $i \in I$.

Therefore $Tu \in \bigcap_{i\in I} U_i$ and hence $\bigcap_{i\in I} U_i$ is T-invar.

Ex5AQ22 **Problem 2** Suppose $T \in \mathcal{L}(V)$ and $\exists$ non zero $v, w \in V$ s.t.

$$Tv = 3w \quad \text{and} \quad Tw = 3v$$

Prove that 3 or -3 is an eigenvalue of T

Sol^n Note that $T(v+w) = Tv + Tw = 3w + 3v = 3(v+w)$

Recall that $\lambda \in \mathbb{F}$ is an eigenvalue of T if $\exists$ NON-ZERO vector $v_0 \in V$ s.t. $Tv_0 = \lambda v_0$. Hence 3 is an eigen value if $v+w \neq 0$. However, if $v+w=0$ then $v-w = (v+w)-2w = -2w \neq 0$ since $w \neq 0$ and $2 \neq 0$. Therefore

$$T(v-w) = Tv - Tw = 3w - 3v = -3(v-w)$$

and -3 is an eigenvalue of T.

Ex 5B Q8 Problem 3 Give an example of $T \in \mathcal{L}(\mathbb{R}^2)$ s.t. $T^4 = -I$.

Sol^n Note that $x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.
It suffices to find $T \in \mathcal{L}(\mathbb{R}^2)$ such that $T^2 - \sqrt{2}T + I = T_0$ the zero transformation.
We may generalize the problem into:
Suppose $\beta = \{e_1, \cdots, e_n\}$ is a basis of a v.sp. V and $p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is a polynomial over $\mathbb{F}$.
Then there exists a linear map $T \in \mathcal{L}(V)$ s.t.
$p(T) = T_0$

Pf: Define $T: V \to V$ by $Te_i = e_{i+1}$ for $i = 1, \cdots, n-1$ and
$Te_n = \sum_{i=1}^{n} -a_{i-1} e_i$ —— $(*)$
We check that $p(T) = T^n + a_{n-1}T^{n-1} + \cdots + a_0 I_V = T_0$
Note $T^i e_1 = e_{i+1}$ for $i = 1, \cdots, n-1$.
Therefore $(*) \Rightarrow T(T^{n-1}e_1) = \sum_{i=1}^{n} -a_{i-1} T^{i-1} e_1$

$\Rightarrow T^n e_1 + \sum_{i=0}^{n-1} a_i T^i e_1 = 0$

$\Rightarrow p(T) e_1 = 0$

Now $p(T) e_i = p(T) T^{i-1} e_1 = T^{i-1} p(T) e_1 = 0$
Therefore by uniqueness $p(T) = T_0$

We have $M(T, \beta) = \begin{bmatrix} 0 & \cdots & \cdots & 0 & -a_0 \\ 1 & \ddots & & \vdots & -a_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -a_{n-1} \end{bmatrix}$ $\left(\det(M(T,\beta) - \lambda I) = (-1)^n p(\lambda) \right)$

For the original problem, we may take the linear transformation T defined by $T(1,0) = (0,1)$ and $T(0,1) = \sqrt{2}(0,1) - (1,0)$

Note that $M(T, \beta) = \begin{bmatrix} 0 & -1 \\ 1 & \sqrt{2} \end{bmatrix} =: A$

Check directly that $A^4 + I = 0$

Friedberg §5.4 Q23

Problem 4 Let T be a linear operator on a fin. dim. v.sp V and W is a T-invar. subspace of V. Suppose that $v_1, \cdots, v_n \in V$ are eigenvectors of T corr. to distinct eigenvalues $\lambda_1, \cdots, \lambda_n \in \mathbb{F}$. Suppose further that $v_1 + \cdots + v_n \in W$. Show that $v_i \in W$ $\forall i$.

Sol^n Use M.I.

The case $n=1$ is tautology ($v_1 \in W \Rightarrow v_1 \in W$).

Assume case $n=k$ is true for some +ve integer k.

For the case $n=k+1$, note that $T(v_1 + \cdots + v_n) = \underbrace{\lambda_1 v_1 + \cdots + \lambda_n v_n}_{\in W}$ since W is T-invar. Also $\lambda_n(v_1 + \cdots + v_n) \in W$

and $(T - \lambda_n I)(v_1 + \cdots + v_n) = (\lambda_1 - \lambda_n)v_1 + \cdots + (\lambda_{n-1} - \lambda_n)v_{n-1} \in W$

Since $\lambda_i - \lambda_n \neq 0$ for $i = 1, \cdots, n-1$ $(\lambda_i - \lambda_n)v_i \neq \vec{0}$ and

$T((\lambda_i - \lambda_n)v_i) = (\lambda_i - \lambda_n) T v_i = \lambda_i (\lambda_i - \lambda_n) v_i$ $\therefore (\lambda_i - \lambda_n)v_i$ is an eigenvector of T

and distinct for $i = 1, \cdots, n-1$. By induction hypothesis

$(\lambda_i - \lambda_n)v_i \in W$ Since $\lambda_i - \lambda_n \neq 0$ $\therefore v_i \in W$ for $i = 1, \cdots, n-1$

$\therefore v_1 + v_2 + \cdots + v_{n-1} \in W$ $\therefore v_n \in W$.

Friedberg §5.4 Q24

Problem 5 If T is a diagonalizable linear operator on a finite-dimensional vector space and W is a nontrivial T-invar. subspace of V, prove that $T|_W$ is a diagonalizable linear operator on W

Sol^n Since T is diagonalizable, if $\lambda_1 \cdots \lambda_n$ are all eigenvalue then $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_n, T)$ — $(*)$

We claim that $W = \bigoplus_{i=1}^{n} (E(\lambda_i, T) \cap W)$ (Note $E(\lambda_i, T) \cap W = E(\lambda_i, T|_W)$.)

$\vdash$ $\sum_{i=1}^{n} E(\lambda_i, T|_W)$ is a direct sum

Since $\sum E(\lambda_i, T)$ is a direct sum

if $v_i \in E(\lambda_i, T) \cap W$ s.t. $\sum_{i=1}^{n} v_i = \vec{0}$ then $v_i = 0 \ \forall i$

$\therefore$ $\sum_{i=1}^{n} E(\lambda_i, T|_W)$ is a direct sum.

$\vdash$ $\sum_{i=1}^{n} E(\lambda_i, T|_W) = W$

"$\subset$" is clear. $\supset$ Let $w \in W$ By $(*)$ $\exists\, v_i \in E(\lambda_i, T)$ s.t. $w = \sum_{i=1}^{n} v_i \in W$. If some $v_i = \vec{0}$ we may ignore them since these v_i belong W already. Hence we may assume the remaining v_i are eigenvectors. By previous problem, $v_i \in W \ \forall i$

$\therefore$ $v_i \in E(\lambda_i, T) \cap W \ \forall i$

Note that if λ is an eigenvalue of $T|_W$ $\exists v \in W$ $v \neq \vec{0}$ s.t. $T|_W(v) = \lambda v$

This v is then also an eigenvector of T and λ is also an eigenvalue of V

Therefore $W = \bigoplus_{i=1}^{n} E(\lambda_i, T|_W)$ is a decomposition of W into eigenspaces. (Some $E(\lambda_i, T|_W)$ may be the trivial subspace).